REFERENCES

1. Slukhaev, V. V., On the geometric theory of stationary motion of fluid. Dokl. Akad. Nauk SSSR, Vol. 196, №3, 1971.
2. Gorbanev, N. N., On certain classes of quasi-stationary flows of a perfect incompressible fluid. PMM Vol. 36, №3, 1972.
3. Biushgens, S. S., Vector field geometry. Izv. Akad. Nauk SSSR, ser. Matem., Vol. 10, №1, 1946.
4. Kochin, N. E., Kibel', I. A. and Roze, N. V., Theoretical Hydrodynamics, Pt. 1. Fizmatgiz, Moscow, 1963.
5. Finikov, S. P., The Kartan Method of External Forms. Gostekhizdat, Moscow, 1948.

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STEADY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID TAKING TEMPERATURE DEPENDENCE OF VISCOSITY INTO ACCOUNT

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We consider the two-dimensional steady flows of a viscous incompressible fluid with viscosity depending exponentially on the temperature. In contrast to the numerical methods for solving this problem [1], we reduce the nonlinear system of equations describing the flow to an infinite sequence of linear equations of elliptic type by means of an expansion in the small parameter appearing in the exponent. We construct a majorizing equation for which the existence of positive solutions guarantees the uniform convergence of iterations on a neighborhood of a zero value of the parameter. As an illustration we study the flow of a viscous fluid in a cylindrical tube with a heat source present.

1. Consider the steady two-dimensional flow of a viscous incompressible fluid with the temperature-dependent viscosity given by the Reynolds relation

$$\mu/\mu_0 = e^{-\alpha T}$$

The system of differential equations of motion, continuity, and energy has the following form [2] upon the introduction of a stream function and omission of the inertial and dissipative terms:

$$\Delta\Delta\psi = 2\alpha\left(\frac{\partial T}{\partial x}\frac{\partial}{\partial x}\Delta\psi + \frac{\partial T}{\partial y}\frac{\partial}{\partial y}\Delta\psi\right) - \left[\alpha^2\left(\frac{\partial T^2}{\partial y} - \frac{\partial T^2}{\partial x}\right) - \alpha\left(\frac{\partial^2 T}{\partial y^2} - \frac{\partial^2 T}{\partial x^2}\right)\right]\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) - 4\left(\alpha^2\frac{\partial T}{\partial x}\frac{\partial T}{\partial y} - \alpha\frac{\partial^2 T}{\partial x\partial y}\right)\frac{\partial^2\psi}{\partial x\partial y} \tag{1.1}$$

$$\Delta T = P\left(\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right), \quad \alpha = \beta\delta T, \quad P = \frac{VL}{a}$$

The geometrical and physical flow parameters are assumed to be dimensionless, being referred to characteristic scaling parameters: the length L, the difference of temperatures δT, and the flow velocity V; β is a parameter which depends on the nature of the fluid and the temperature range; a is the constant coefficient of thermal diffusivity. Let the flow domain be an open connected set Ω of Euclidean space with a piecewise-smooth boundary Γ. We can then take the boundary conditions in the following form:

$$\psi\,|_\Gamma = c, \qquad \frac{\partial\psi}{\partial n}\Big|_\Gamma = 0, \qquad T\,|_\Gamma = T(\Gamma) \tag{1.2}$$

Here $\partial\psi / \partial n$ is the normal derivative of the stream function, c is an arbitrary constant, and $T(\Gamma)$ is a function of a point on the domain boundary. In the case of unbounded domains it is necessary to impose certain conditions [3] on the temperature and the stream function.

Let us assume that the functions $\psi(x, y, \alpha)$ and $T(x, y, \alpha)$ are represented by the formal expressions

$$\psi(x, y, \alpha) = \sum_{k=0}^{\infty} \psi_k(x, y)\,\alpha^k, \qquad T(x, y, \alpha) = \sum_{k=0}^{\infty} T_k(x, y)\,\alpha^k \tag{1.3}$$

Substituting the relations (1.3) into the system of equations (1.1), we obtain an infinite system of equations of elliptic type

$$\Delta\Delta\,\psi_k = F_k(\psi_0, \psi_1, \ldots, \psi_{k-1}, T_0, T_1 \ldots T_{k-1}) \tag{1.4}$$

$$\Delta T_k = P\left\{\left(\frac{\partial\psi_0}{\partial y}\frac{\partial T_k}{\partial x} - \frac{\partial\psi_0}{\partial x}\frac{\partial T_k}{\partial y}\right) + V_k(\psi_1, \ldots \psi_k, T_0, \ldots, T_{k-1})\right\}$$

$$F_k = -\sum_{m=0}^{k-2}\ \sum_{n=0,\ m+n+l=k-2}^{k-2} 4\frac{\partial T_m}{\partial x}\frac{\partial T_n}{\partial y}\frac{\partial^2\psi_l}{\partial x\partial y} + \left(\frac{\partial T_m}{\partial y}\frac{\partial T_n}{\partial y} - \frac{\partial T_m}{\partial x}\frac{\partial T_n}{\partial x}\right)\times \tag{1.5}$$

$$\left(\frac{\partial^2\psi_l}{\partial y^2} - \frac{\partial^2\psi_l}{\partial x^2}\right) + \sum_{m=0,\ m+l=k-1}^{k-1} 2\left(\frac{\partial T_m}{\partial x}\frac{\partial}{\partial x}\Delta\psi_l + \frac{\partial T_m}{\partial y}\frac{\partial}{\partial y}\Delta\psi_l\right) +$$

$$\left(\frac{\partial^2 T_m}{\partial y^2} - \frac{\partial^2 T_m}{\partial x^2}\right)\left(\frac{\partial^2\psi_l}{\partial y^2} - \frac{\partial^2\psi_l}{\partial x^2}\right) + 4\frac{\partial^2 T_m}{\partial x\partial y}\frac{\partial^2\psi_l}{\partial x\partial y}$$

$$V_k = \sum_{m=0,\ l\geqslant 1,\ m+l=k}^{k-1}\left(\frac{\partial T_m}{\partial x}\frac{\partial\psi_l}{\partial y} - \frac{\partial T_m}{\partial y}\frac{\partial\psi_l}{\partial x}\right)$$

In seeking the k-th approximation we take the functions F_k and V_k as known.

When the expansions (1.3) are taken into account, the boundary conditions (1.2) become

$$\psi_0\,|_\Gamma = c, \quad \frac{\partial\psi_0}{\partial n}\Big|_\Gamma = 0, \quad T_0\,|_\Gamma = T(\Gamma) \tag{1.6}$$

$$\psi_k\,|_\Gamma = 0, \quad \frac{\partial\psi_k}{\partial n}\Big|_\Gamma = 0, \quad T_k\,|_\Gamma = 0, \quad k \geqslant 1$$

Thus, to solve the problem it is necessary to integrate nonhomogeneous differential equations of elliptic type with the known boundary conditions (1.6).

2. We turn now to a clarification of the problem concerning the uniform convergence of the expansions (1.3) in some interval of variation of the parameter α. If we assume boundedness of the flow domain and suitable smoothness of the function $T(\Gamma)$, the stream function and the temperature, and their derivatives to the third and second order inclusive, will be bounded and continuous at $\alpha = 0$. Then in the domain $\Omega + \Gamma$ all the succeeding k-approximations will possess this property.

Let

$$\max | F_k (x, y) | \leqslant R_k, \quad \max | V_k (x, y) | \leqslant H_k \quad x, y \in \Omega + \Gamma \tag{2.1}$$

It is obvious that we can construct sequences of positive numbers h_k and d_k such that

$$\left| \frac{\partial^i \psi_k}{\partial x^m \partial y^{i-m}} \right| \leqslant R_k h_k, \quad i = 0, 1, 2, 3, \quad m = 1, 2, 3 \tag{2.2}$$
$$\left| \frac{\partial^i T_k}{\partial x^m \partial y^{i-m}} \right| \leqslant H_k d_k, \quad i = 0, 1, 2, \quad m = 1, 2$$

It is known from the theory of elliptic equations [3] that the sequence of positive numbers h_k is bounded from above. We shall assume that the sequence d_k possesses the same property. We note that the assumption made is invalid in the general case (for the second derivatives of the functions $T_k (x, y)$), however, there exist a number of practically important problems (flow in tubes, diffusors) for which boundedness of the sequence d_k can be proved on the basis of analysis of ordinary differential equation systems. Thus

$$h_k \leqslant h, \quad d_k \leqslant d, \quad k = 0, 1, 2, \ldots \tag{2.3}$$

Taking the relations (2.3) into account, we can strengthen the inequalities (2.2). Thus

$$\left| \frac{\partial^i \psi_k}{\partial x^m \partial y^{i-m}} \right| \leqslant R_k h = U_k, \quad \left| \frac{\partial^i T_k}{\partial x^m \partial y^{i-m}} \right| \leqslant H_k d = \theta_k \tag{2.4}$$

As R_k and H_k in the inequalities (2.1) we can take the numbers

$$R_k = 8S_1 + 16S_2, \quad H_k = 2PS_3 \tag{2.5}$$
$$S_1 = \sum_{m=0}^{k-2} \sum_{n=0}^{k-2} \theta_m \theta_n U_l \quad (m + n + l = k - 2)$$
$$S_2 = \sum_{m=0}^{k-1} \theta_m U_l \quad (m + l = k - 1), \quad S_3 = \sum_{m=0}^{k-1} \theta_m U_l \quad (m + l = k, \ l \geqslant 1)$$

Let us form the following series:

$$U (\alpha) = \sum_{k=0}^{\infty} \alpha^k U_k, \quad \theta = \sum_{k=0}^{\infty} \alpha^k \theta_k \tag{2.6}$$

Taking the relations (2.4) and (2.5) into account, we find the following expressions for the general terms of these series:

$$U_k = h (8S_1 + 16S_2), \quad \theta_k = 2dPS_3 \tag{2.7}$$

By replacing in the series (2.6) all the terms, except the zero terms, by their expressions from the Eqs. (2.7), we arrive at the relations

$$U = U_0 + 16h\alpha U\theta + 8h\alpha^2 U\theta^2 \tag{2.8}$$
$$\theta = \theta_0 + 2dP \ (U - U_0) \ \theta$$

After simplifying, we obtain the equation

$$\Phi (\alpha, W) = \sum_{i=1}^{4} p_i (\alpha) W^{4-i} (\alpha) = 0 \quad (W = U\theta) \tag{2.9}$$

$$p_1(\alpha) = 32hd^2l^2P^2\alpha^2, \quad p_2(\alpha) = 32hdl^2\theta_0P\alpha^2 + 32hdlP\alpha$$
$$p_3(\alpha) = 8hl^2\theta_0^2\alpha^2 + 16hl\theta_0\alpha + 2dlU_0P - 1$$
$$p_4(\alpha) = U_0\theta_0 l, \qquad l = (1 + 2dU_0P)^{-1}$$

This equation is a majorant for the functions (1.3); the existence of positive solutions of (2.9) for $0 \leqslant \alpha \leqslant \alpha^*$ implies the uniform convergence of the functions $\psi(x, y, \alpha)$ and $T(x, y, \alpha)$ and their derivatives in the domain $\Omega + \Gamma$ under consideration. For this case Eq. (2.9) is algebraic in the function $W(\alpha)$ and, consequently, the radius of convergence is determined after W has been eliminated from the system of equations

$$\Phi(\alpha^*, W) = 0, \quad \partial\Phi(\alpha^*, W) / \partial W = 0$$

When $\alpha \ll 1$, we have, upon neglecting the α^2 terms in comparison with the α terms in the Eqs. (2.9), the following expression for the radius of convergence of the series (2.6):

$$\alpha^* = \frac{m - \sqrt{m^2 - 1}}{16h\theta_0}, \qquad m = 1 + 4dU_0P$$

3. As an example we consider the flow of a viscous fluid in a cylindrical tube of diameter $2R$. We assume that at each point of the flow domain there is a constant heat source of strength Q; in addition, we assume the tube walls to be maintained at the zero temperature. In this case flow of the fluid in parallel jets turns out to have no influence on the temperature distribution which is expressed by the following formula [4]:

$$T = \frac{\alpha}{\beta}(1 - r^2), \qquad \alpha = \frac{\beta Q R^2}{4\lambda} \tag{3.1}$$

where r is the dimensionless tube radius and λ is the thermal conductivity coefficient. The equation of motion in dimensionless form is given by

$$\frac{d}{dr}\Delta v + 4\alpha^2 r^2 \frac{dv}{dr} + 4\alpha r \frac{d^2v}{dr^2} + 4\alpha \frac{dv}{dr} = 0 \tag{3.2}$$

We assume that the velocity of the fluid is representable as a series in increasing powers of α

$$v(r, \alpha) = \sum_{k=0}^{\infty} v_k(r)\alpha^k \tag{3.3}$$

Substituting the relation (3.3) into Eq. (3.2) and equating terms with like powers of α, we obtain an equation for the k-th approximation in which the right side of the equation is known

$$\frac{d}{dr}\Delta v_k = F_k(v_1, v_2, \ldots v_{k-1}) \tag{3.4}$$

$$F_k(v_1, v_2, \ldots v_{k-1}) = -4r^2\frac{dv_{k-2}}{dr} - 4r\frac{d^2v_{k-1}}{dr^2} - 4\frac{dv_{k-1}}{dr}$$

We take the boundary conditions in the usual form

$$v_k(1) = 0 \quad (k = 0, 1, 2 \ldots), \qquad \int_0^1 rv_0(r)\,dr = \frac{1}{2} \tag{3.5}$$
$$\int_0^1 rv_k(r)\,dr = 0 \qquad (k \geqslant 1)$$

On the basis of two approximations we determine the flow velocity to be

$$v(r) = 2(1 - r^2) - \alpha/3\,(1 - r^2)(3r^2 - 1)$$

The convective terms in the energy equation are identically zero, so that the majorizing equation assumes the particularly simple form

$$(\alpha^2 + 2\alpha - {}^9/_{80})\, W + {}^9/_{20} = 0 \qquad (3.6)$$

For the existence of positive solutions of Eq. (3.6) it is necessary to consider $\alpha < \alpha^* = 0.055$. The value $\alpha^* = 0.055$ determines the radius of convergence of the series for the velocity and for its first two derivatives; it is obvious that the radius can be increased, but at the expense of a refinement in the estimate for α^*.

Convergence of the expansion (3.3) can also be established through a direct verification of the identity

$$v_k(r) = \frac{1}{k!} \frac{d^k v(r, \alpha)}{d\alpha^k}\Bigg|_{\alpha=0}$$

$$v(r, \alpha) = \frac{\alpha}{\exp \alpha - (\alpha + 1)} \{\exp [\alpha(1 - r^2)] - 1\}$$

Here $v(r, \alpha)$ is the exact solution of Eq. (3.2).

REFERENCES

1. Dzhakupov, K. B., Calculation of two-dimensional hydrodynamic flows and heat transfer of a viscous fluid. Izv. Sibirsk. Otdel. Akad. Nauk SSSR, Ser. Tekhn. Nauk, Vol. 1, № 3, 1972.
2. Targ, S. M., Fundamental Problems of Laminar Flow Theory. Gostekhizdat, Moscow-Leningrad, 1951.
3. Miranda, C., Partial Differential Equations of Elliptic Type. 2nd Rev. Ed., Springer-Verlag, New York, 1969.
4. Gröber, H., Erk, S. and Grigull, U., Fundamental Studies in Heat Transfer (Russian translation), Moscow, 1958.

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ON THE EFFECT OF INJECTION ON BOUNDARY LAYER SEPARATION

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An integral inequality is obtained for the rate of fluid injection into the boundary layer of a streamlined surface. Separation takes place when this inequality is satisfied and the pressure gradient is nonnegative. In particular, separation occurs whenever the positive injection rate is constant, independently of the magnitude of that rate. The results obtained in [1] where it was shown that separation takes place at sufficiently high injection rates, are thus refined.

1. We consider the system

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{dp}{dx}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1.1)$$